

q-Deformed de Sitter/Conformal Field Theory Correspondence

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Abstract

Unitary principal series representations of the conformal group appear in the dS/CFT correspondence. These are infinite dimensional irreducible representations, without highest weights. In earlier work of Güijosa and the author it was shown for the case of two-dimensional de Sitter, there was a natural q -deformation of the conformal group, with q a root of unity, where the unitary principal series representations become finite-dimensional cyclic unitary representations. Formulating a version of the dS/CFT correspondence using these representations can lead to a description with a finite-dimensional Hilbert space and unitary evolution. In the present work, we generalize to the case of quantum-deformed three-dimensional de Sitter spacetime and compute the entanglement entropy of a quantum field across the cosmological horizon.

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I. INTRODUCTION

Despite much progress in understanding string theory in backgrounds with negative cosmological constant, the study of the more physically relevant case of positive cosmological constant is still in its infancy. One proposal for formulating quantum gravity in de Sitter space has been proposed in [1]. See also [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] for further developments and background. This proposal involves formulating a holographic dual to de Sitter, with the isometry group of de Sitter identified with the conformal group of the dual field theory.

At first sight this proposal seems to conflict with the idea that de Sitter space should be described by a theory with a finite-dimensional Hilbert space, bounded by $e^{S_{BH}}$, with S_{BH} the Bekenstein-Hawking entropy associated with the cosmological horizon [21, 22]. The unitary representations of the conformal group have continuous unbounded energy spectra, so even the single particle Hilbert space is infinite-dimensional.

In recent works new versions of the dS/CFT correspondence have been proposed built on finite-dimensional Hilbert spaces. In [23] it is suggested the Hilbert space be built out of fermionic degrees of freedom which transform under the $SO(d - 1) \times \mathbb{R}$ symmetry that leaves the horizon of a comoving observer fixed. It is argued a S-matrix invariant under the full de Sitter isometries may nevertheless be constructed.

In the present paper we further elaborate on the alternative proposal of [24] where the de Sitter isometry group is replaced with a quantum deformed version. See [25, 26] for earlier related work. In the original work [24] the proposal was made in its most detailed form for the case of two-dimensional de Sitter space, where the isometry group is $SL(2, \mathbb{R})$ which is deformed to $SL_q(2, \mathbb{R})$ with $q = e^{2\pi i/N}$. Cyclic unitary irreducible representations of the quantum group were found to become unitary principal series representations of the conformal group as $N \rightarrow \infty$. The construction agrees with the proposal of Strominger in this limit, but can yield a finite-dimensional Hilbert space for N finite. These representations form natural building blocks for a new formulation of dS/CFT that avoids problems associated with non-unitarity and continuous spectra. One question left unanswered in [24] was the detailed relation between N and the cosmological constant. This was related to the fact that the Bekenstein-Hawking entropy formula is degenerate in two-dimensions, where the horizon is a point.

In the present work we generalize the construction of [24] to three-dimensional de Sitter, with a view to obtaining a better understanding of the Bekenstein-Hawking entropy in this framework. We find it is necessary to modify the proposal for the inner product, in order to obtain unitary representations of the quantum isometry group. Nevertheless, these representations deform to unitary principal series representations of the de Sitter isometry group $SL(2, \mathbb{C})$ in the $N \rightarrow \infty$ limit, where the construction coincides with the original proposal of Strominger.

We perform the analog of the 't Hooft brick-wall entropy calculation of the horizon entropy for the quantum deformed three-dimensional de Sitter space. The result is regulated for finite N , and here we are able to relate the cosmological constant to the parameters that characterize the representations of the quantum isometry group. In future work we hope to extend these results to a quantum deformation of the full Virasoro algebra, which can potentially yield a complete description of the entropy of de Sitter space.

II. BASIC SETUP

Three-dimensional de Sitter space can be viewed as a hyperboloid embedded in flat Minkowski space (t, x, y, z)

$$-t^2 + x^2 + y^2 + z^2 = R^2 ,$$

and so inherits the isometry group $SO(3, 1) \approx SL(2, \mathbb{C})$. For generic masses larger than the Hubble scale, the representations of the isometry group relevant for the dS/CFT correspondence are the unitary principal series representations [16, 24]. These representations are described in detail in [27, 28]. They can be realized as smooth functions on the complex plane with asymptotic behavior

$$f(z, \bar{z})|_{|z| \rightarrow \infty} \sim z^{2\tau} \bar{z}^{2\tilde{\tau}} , \quad (1)$$

with $\tau = \frac{s}{4} + \frac{i\rho}{4} - \frac{1}{2}$ and $\tilde{\tau} = -\frac{s}{4} + \frac{i\rho}{4} - \frac{1}{2}$, with s integer and ρ real, and which have finite norm with respect to the inner product

$$(f, g) = \int d^2z \overline{g(z, \bar{z})} f(z, \bar{z}) . \quad (2)$$

This is the norm induced on the boundary by the standard Klein-Gordon norm in the bulk [24]. A generator of $SL(2, \mathbb{C})$ can be written

$$c_n L_n + \bar{c}_n \bar{L}_n , \quad (3)$$

where \bar{c}_n are the complex conjugates of the c_n with $n = -1, 0, 1$. The L_n satisfy an $sl(2, \mathbb{R})$ algebra. Likewise the \bar{L}_n satisfy an $sl(2, \mathbb{R})$ algebra and commute with the L_n . Acting on the function $f(z, \bar{z})$ the generators may be written

$$\begin{aligned} L_{-1} &= \partial_z , \quad L_0 = -z\partial_z + \tau , \quad L_1 = -z^2\partial_z + 2\tau z \\ \bar{L}_{-1} &= \partial_{\bar{z}} , \quad \bar{L}_0 = -\bar{z}\partial_{\bar{z}} + \tilde{\tau} , \quad \bar{L}_1 = -\bar{z}^2\partial_{\bar{z}} + 2\tilde{\tau}\bar{z} . \end{aligned}$$

It is important to note that basis functions that diagonalize L_0 and \bar{L}_0 such as $z^\alpha \bar{z}^\beta$ lie outside the space of functions of finite norm. If this were not the case, it would be easy to show the representation was not unitary, by considering the norm of, for example, $(L_{-1} + \bar{L}_{-1})z^\alpha \bar{z}^\beta$. However provided we restrict to functions of finite norm, the action of the generators maintain this condition, and the principal series representation is indeed unitary.

According to the dS/CFT proposal, the parameters that appear in this representation are related to the mass m and spin s of fields in the de Sitter background. Usually only scalar fields ($s = 0$) are considered, where

$$\rho = \sqrt{m^2 - 1} .$$

The bulk scalar field is dual to an operator in the CFT with conformal dimension

$$h_- = -2\tau - 2\tilde{\tau} - 1 = 1 - i\rho .$$

However there is an equivalence [27, 29] of the representations under $\tau \rightarrow -1 - \tau$, $\tilde{\tau} \rightarrow -1 - \tilde{\tau}$, for generic values of ρ , so the same representation also describes operators with conformal weight

$$h_+ = 1 + i\rho .$$

One obtains the bulk field configuration by convolving the boundary field configuration on past asymptotic infinity of de Sitter with the bulk to boundary propagator. Due to the equivalence relation, only a single representation of the boundary theory is needed to describe both these operators. The Bunch-Davies vacuum is expected to be the stable vacuum of de Sitter [18], which selects a particular linear combination of these operators as possible sources in the boundary theory. This corresponds to a specific choice of bulk to boundary propagator. See [24] for further discussion of these points.

In terms of the canonical basis of $su(1, 1) \approx sl(2, \mathbb{R})$ generators that we will use below, the L_n generators are

$$\begin{aligned} L_0 &= \frac{1}{2}(X_+ + X_-) \\ L_{-1} &= \frac{i}{2}(X_+ - X_- - H) \\ L_1 &= \frac{i}{2}(X_- - X_+ - H) . \end{aligned} \tag{4}$$

The X_{\pm}, H notation is typical in the literature on quantum groups [30, 31]. The *-structure inherited from (2) acts via

$$\begin{aligned} X_{\pm} &\rightarrow -\bar{X}_{\mp}, & H &\rightarrow \bar{H} \\ \bar{X}_{\pm} &\rightarrow -X_{\mp}, & \bar{H} &\rightarrow H \end{aligned} \tag{5}$$

or equivalently

$$\begin{aligned} L_n &\rightarrow -\bar{L}_n \\ \bar{L}_n &\rightarrow -L_n . \end{aligned}$$

Another *-structure that will play an important role in the following acts without the left-right exchange as

$$\begin{aligned} X_{\pm} &\rightarrow -X_{\mp}, & H &\rightarrow H \\ \bar{X}_{\pm} &\rightarrow -\bar{X}_{\mp}, & \bar{H} &\rightarrow \bar{H} \end{aligned} \tag{6}$$

so that

$$\begin{aligned} L_n &\rightarrow -L_n \\ \bar{L}_n &\rightarrow -\bar{L}_n . \end{aligned}$$

This is equivalent to changing the inner product from (2) to

$$(f, g) = \int d^2z \overline{g(Pz, P\bar{z})} f(z, \bar{z}) = \int d^2z \overline{g(\bar{z}, z)} f(z, \bar{z}) \tag{7}$$

where we have defined the parity operation as $Pz = \bar{z}$. We will discuss this inner product in more detail below.

III. q -DEFORMED dS_3/CFT_2

As we will see a generalization of [24] to this case is not entirely straightforward, owing to the fact that the isometry group $SO(3, 1)$ (or more precisely its complexification) is not simple. This means there is not a unique q -deformation, but rather multi-parameter families of q -deformation [32]. We will consider a quantum deformation of $sl(2, \mathbb{C})$ that preserves the left-right factorization of the group elements and replaces the $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$ algebras by $U_q(su(1, 1)_\mathbb{R}) \times U_q(su(1, 1)_\mathbb{R})$, using the same notation as [24]. We will explain this in more detail momentarily[43]. It is worth noting this is not the usual q -deformation that has been considered in much of the literature on the the quantum deformed Lorentz group , where also q is usually taken to be real, rather than a root of unity, and where the quantum deformation is constructed to maintain an $su_q(2)$ subalgebra.

Our basic building block will be the quantum group $SL_q(2, \mathbb{C})$, or more precisely, the universal enveloping algebra $U_q(sl(2, \mathbb{C}))$ with complex coefficients built out of generators K, X_+, X_- satisfying [31]

$$KK^{-1} = K^{-1}K = 1, \quad KX_\pm K^{-1} = q^{\pm 2}X_\pm, \quad [X_+, X_-] = \frac{K - K^{-1}}{q - q^{-1}}. \quad (8)$$

The universal enveloping algebra consists of the space spanned by the monomials

$$(X_+)^n K^m (X_-)^l \quad (9)$$

with $m \in \mathbb{Z}$ and n, l non-negative integers. The generator K can be identified with q^H , but it is important to keep in mind the enveloping algebra is defined in terms of products of K rather than H . The universal enveloping algebra is a Hopf algebra with comultiplication Δ defined as

$$\begin{aligned} \Delta(K) &= K \otimes K \\ \Delta(X_+) &= X_+ \otimes K + 1 \otimes X_+ \\ \Delta(X_-) &= X_- \otimes 1 + K^{-1} \otimes X_- \end{aligned}$$

and with antipode S and counit ε defined as

$$\begin{aligned} S(K) &= K^{-1}, \quad S(X_+) = -X_+K^{-1}, \quad S(X_-) = -KX_-, \\ \varepsilon(K) &= 1, \quad \varepsilon(X_\pm) = 0. \end{aligned}$$

For the special case that q is a root of unity, we can extract a real subalgebra of this Hopf algebra, that we denote $U_q(su(1, 1)_{\mathbb{R}})$. This is done by defining a map

$$\theta(X^{\pm}) = -X^{\pm}, \quad \theta(K) = K^{-1}, \quad (10)$$

and showing that the restriction $a^* = \theta(a)$ where $a \in U_q(su(1, 1))$ and $*$ is defined in (6), is compatible with the algebra and comultiplication structure.

The full q -deformed $sl(2, \mathbb{C})$ is then defined as two mutually commuting left-right copies of $U_q(su(1, 1)_{\mathbb{R}})$, with generators X_{\pm} , K and \bar{X}_{\pm} , \bar{K} . In the classical case, $sl(2, \mathbb{C})$ is generated by elements of the form (3). The corresponding statement in the q -deformed case is that we allow for elements A of $U_q(su(1, 1)_{\mathbb{R}}) \times U_q(su(1, 1)_{\mathbb{R}})$ with complex coefficients, but then we impose a restriction with respect to the real form that acts on the generators as

$$\phi(L_n) = \bar{L}_n, \quad \phi(\bar{L}_n) = L_n$$

together with complex conjugation of numerical coefficients, such that $\phi(A) = A$. This is equivalent to the action

$$\begin{aligned} X_{\pm} &\rightarrow \bar{X}_{\mp}, \quad H \rightarrow -\bar{H}, \quad K \rightarrow \bar{K} \\ \bar{X}_{\pm} &\rightarrow X_{\mp}, \quad \bar{H} \rightarrow -H, \quad \bar{K} \rightarrow K. \end{aligned} \quad (11)$$

To be specific, this is an involutive morphism, and co-morphism (i.e. the order of generators is not changed, and the order in the co-product also remains unchanged). We denote the final algebra $U_q(SL(2, \mathbb{C})_{\mathbb{R}})$ where the \mathbb{R} denotes this is a real algebra (i.e. one with six independent generators, versus, say, $U_q(SL(2, \mathbb{C}))$ in standard notation in the quantum group literature [30, 31], that has three independent generators with arbitrary complex coefficients).

In order for a representation to be unitary, any state acted on by products of the generators (3) should be positive, semi-definite, vanishing only for the trivial state. For q a root of unity, irreducible representations are finite dimensional [31], so one can always pick an orthonormal basis where K and \bar{K} are diagonal. It is easy see that the state $(X_+ + \bar{X}_-)|\psi\rangle$ where $|\psi\rangle$ is any basis element, has vanishing norm with respect to the $*$ -structure (5). Therefore there are no nontrivial unitary representations of the q -deformed algebra with respect to (5). Instead we will construct a non-unitary representation that goes over to a unitary principal series representation at $q = 1$. Furthermore we will see this non-unitary representation is actually unitary with respect to the different $*$ -structure (6).

The irreps we have in mind are constructed from a left-right tensor product of two cyclic representations of the type discussed in [24] in the context of dS_2 . For q a root of unity (we will take $q = e^{2\pi i/N}$ with N odd) there exists a class of finite-dimensional irreducible representations of the quantum group $U_q(SL(2, \mathbb{C}))$ that can be realized on the N -dimensional basis $|m\rangle$ with $m = 0, \dots, N - 1$ [31] and parametrized by the complex numbers a, b, λ :

$$\begin{aligned} K|m\rangle &= q^{-2m}\lambda|m\rangle \\ X_+|m\rangle &= \left(ab + \frac{q^m - q^{-m}}{q - q^{-1}} \frac{\lambda q^{1-m} - \lambda^{-1} q^{m-1}}{q - q^{-1}}\right) |m-1\rangle \\ X_-|m\rangle &= |m+1\rangle \end{aligned}$$

supplemented by the additional cyclic operations

$$X_+|0\rangle = a|N-1\rangle, \quad X_-|N-1\rangle = b|0\rangle, \quad (12)$$

and the consistency condition

$$|b|^2 = - \prod_{j=0}^{N-1} s(j) \quad (13)$$

where $s(j) \equiv ab + \frac{q^m - q^{-m}}{q - q^{-1}} \frac{\lambda q^{1-m} - \lambda^{-1} q^{m-1}}{q - q^{-1}}$. For $a, b \neq 0$ there are no highest- or lowest-weight states and the representation is called cyclic. As in [24] we will be interested in the case where we set $\lambda = q^{N-1}$, and let us set

$$ab = \tau(\tau + 1) - \frac{1}{4}(N^2 - 1)$$

which fixes a and b up to a mutual phase, when combined with (13). In the limit $N \rightarrow \infty$, it was shown that this becomes a principal series representation of $sl(2, \mathbb{R})$, characterized by the complex parameter τ .

We build our candidate representation of the full $U_q(sl(2, \mathbb{C})_{\mathbb{R}})$ using a left-right tensor product of two of these representations with

$$\tau = -\frac{1}{2} + \frac{i\rho}{4} + \frac{s}{4}, \quad \tilde{\tau} = -\frac{1}{2} + \frac{i\rho}{4} - \frac{s}{4} \quad (14)$$

with ρ real and s integer as above. To check unitarity, it suffices to check that $(c_n L_n + \bar{c}_n \bar{L}_n)|\psi\rangle$ has positive norm for all choices of the complex coefficients c_n with respect to the *-structure (6). Converting to $U_q(su(1, 1)_{\mathbb{R}})$ generators using (4), we must check that

$$(\bar{c}_0 H - c_0 \bar{H})(c_0 H - \bar{c}_0 \bar{H}) \geq 0, \quad -(\bar{c}_+ X_- + c_+ \bar{X}_+)(c_+ X_+ + \bar{c}_+ \bar{X}_-) > 0, \text{ and } -(\bar{c}_- X_+ + c_- \bar{X}_-)(c_- X_- + \bar{c}_- \bar{X}_+) > 0.$$

Using the fact that m and \bar{n} are real, the first inequality implies

$$\langle m, \bar{n} | (\bar{c}_0 H - c_0 \bar{H})(c_0 H - \bar{c}_0 \bar{H}) | m, \bar{n} \rangle = |c_0 m - \bar{c}_0 \bar{n}|^2 \langle m, \bar{n} | m, \bar{n} \rangle \geq 0$$

which is trivially satisfied. Next we consider the second inequality, which reduces to

$$|c_+|^2 \langle m, \bar{n} | - X_- X_+ - \bar{X}_+ \bar{X}_- | m, \bar{n} \rangle > 0.$$

The results of [24] can then be used to reduce this expression to the inequality

$$-\tau(\tau + 1) - \tilde{\tau}(\tilde{\tau} + 1) > 0$$

which holds assuming the form (14). No new conditions emerge from considering the final inequality.

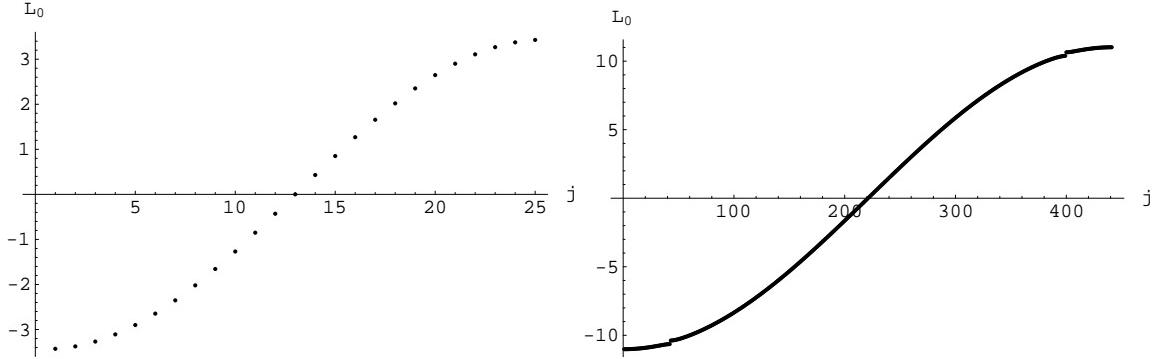
In the limit $N \rightarrow \infty$, the representation becomes a principal series representation of the individual left or right $sl(2, \mathbb{R})$ factors. Taking into account the elements of the $sl(2, \mathbb{C})$ algebra are built out of these $sl(2, \mathbb{R})$ generators with complex coefficients (3), we obtain a principal series representation of $sl(2, \mathbb{C})$ characterized by (ρ, s) as described in the previous section. This is unitary with respect to the norm (2) at $q = 1$, but non-unitary for $q \neq 1$. On the other hand, with respect to the new norm appropriate for the *-structure (6) the representation is unitary for general q a root of unity.

We therefore conjecture this new norm emerges from a non-perturbative formulation of gravity in a de Sitter background. This bears some similarity to Witten's inner product [33], which instead introduces the operation of CPT , and generates the adjoint map $L_n \rightarrow L_{-n}$ typical of conformal field theory. Following similar logic as [33] we conjecture the inner product (7) gives the appropriate unitary norm for $q = e^{2\pi i/N}, q \neq 1$. Having to make this different choice for the inner product in the quantum version of de Sitter, versus the classical $q = 1$ case, is not entirely unreasonable, since we expect the semiclassical limit of quantum de Sitter to be rather subtle. In particular, the finite N theory includes modes that become non-normalizable as $N \rightarrow \infty$, so it is not surprising the standard norm (2) is non-unitary at finite N . Conversely, while the new inner product is unitary at finite N , it is no longer unitary at $N = \infty$, where it takes the form (7), assuming we impose the standard finiteness and fall-off conditions (1).

However we expect that with a proper definition of the bulk to boundary map at finite N , it should be possible to see the new norm reproduces ordinary perturbative quantum field theory in a de Sitter background as $N \rightarrow \infty$. In effect, the extra parity operation can be undone in the formulation of the finite N bulk to boundary map. We hope to explore this further in future work, however it is first necessary to develop further q -deformed de Sitter geometry, and the harmonic analysis on this space. This has not been studied for q a root of unity, however for q real substantial progress has been made in [34]. There a q -deformed version of three-dimensional Lobachevsky space is described, and analogs of the Bessel functions and Macdonald functions are defined. We have checked that the quantum Lobachevsky space can be analytically continued to describe a version of q -de Sitter space at q a root of unity, however further generalization is necessary to obtain coordinates that yield the cyclic representations of the quantum de Sitter group needed in the present context. We can at least offer a more precise description of the geometry of the quantum plane where the holographic CFT lives. We present this in the appendix and will use these results in the following section when we compute the entanglement entropy on de Sitter.

In [24] it was pointed out that if one changes basis so that L_0 is diagonal rather than K , then the $N \rightarrow \infty$ limit does not smoothly match that of the principal series representation of $SL(2, \mathbb{R})$, where L_0 has continuous unbounded imaginary eigenvalues. Instead one has a spectrum where the eigenvalues of L_0 range over approximately $-iN/2$ to $iN/2$ with imaginary spacings of order 1. This is also an issue for the situation we have described above. We emphasize, in the basis where K and \bar{K} are diagonal that limit $N \rightarrow \infty$ does correctly reproduce the principal series representation, with each weight appearing with unit degeneracy as required. This is one of many subtleties dealing with infinite-dimensional representations. For the sake of definiteness we will refer to these representations as Type I.

It is worth pointing out that by including M^2 different representations of $U_q(sl(2, \mathbb{C})_{\mathbb{R}})$ it is possible to obtain the principal series representation as $N \rightarrow \infty$ even in the basis where L_0 and \bar{L}_0 are diagonal. The point is that the eigenvalues of L_0 depend on the phase of a (and likewise for the right-moving factor). If we take a direct sum of M different irreducible representations $a_k = |a|e^{2\pi i k/M}$, with $k = 0, \dots, M - 1$ we obtain a spectrum for L_0 with typical spacing of order i/M . Combining this with the right moving factor, we can then build a reducible representation of $U_q(sl(2, \mathbb{C})_{\mathbb{R}})$ built out of a direct sum of M^2 different irreducible $(a_k, b_k, \tilde{a}_l, \tilde{b}_l)$ representations with $k, l = 0, \dots, M - 1$, which has a continuum



Spectrum of L_0 in the reducible representation composed of a direct sum of irreducible representations with $a_k = |a|e^{2\pi ik/N}$ for $k = 0, \dots, N - 1$. The values $N = 5$, and $N = 21$ are shown. The index j labels the N^2 different eigenvalues in the reducible representation.

Figure 1:

spectrum for L_0 and \bar{L}_0 as $N \rightarrow \infty$, provided we also send $M \rightarrow \infty$. The simplest solution is to set $M = N$, but other versions of the limit may also be worth exploring. Figure 1 illustrates the approach to a continuous spectrum as $N = M$ is increased[44]. We will refer to this class of representations as Type II.

Let us comment on some of the physical characteristics of these representations. As discussed in [16, 24], $(L_0 + \bar{L}_0)/l$ generates time translations for a static patch observer. Here we have restored factors of the Hubble radius l . For classical de Sitter space with Lorentzian signature, the spectrum of this generator is continuous and unbounded above and below.

However the Type I q -deformed representations contain built in infrared and ultraviolet cutoffs, $E_{IR} = 1/l$, $E_{UV} = N/l$. It is natural to identify the ultraviolet cutoff with the Planck scale (for want of any other interesting UV scales in the problem), which provides us with a way to fix $N = m_{pl}l$. Since the infrared cutoff is fixed in terms of the Hubble scale, this points to a new discreteness in the energy levels of quantum gravity in de Sitter space. It is interesting to note $E_{IR} = 10^{-34}eV$, too small to be accessible in present day experiments. But the idea that quantum gravity could be tested by performing ultra-low temperature experiments is intriguing. It would be very interesting to develop other experimental tests of the q -deformation model of de Sitter.

Type II representations, on the other hand, contain cutoffs of the form $E_{IR} = 1/Nl$, and

$E_{UV} = N/l$. For large N , these representations approach the continuous classical spectrum.

IV. ENTANGLEMENT ENTROPY

't Hooft introduced the brick wall entropy model [35] to provide a simple toy model that naturally explains why the quantum properties of a black hole are associated with its horizon. In this model one places an ultraviolet cutoff on a field such that it vanishes inside some fixed radial distance from the horizon. This cutoff explicitly breaks general coordinate invariance. It is also necessary to introduce an infrared cutoff to regulate the usual volume divergence of the entropy associated with a field at finite temperature. 't Hooft noticed that as one removes the ultraviolet cutoff, a divergence appears proportional to the horizon area. The entire horizon entropy of the black hole can then be accounted for by adjusting the ultraviolet cutoff and the number of species of field. Subsequent work using analogous calculations in flat spacetime [36, 37, 38, 39, 40] emphasized the interpretation as entanglement entropy between degrees of freedom inside and outside the horizon. The brick wall model has been generalized to three-dimensional de Sitter space in [41].

The free energy depends on the statistics of particles under consideration. The results for the free energy are the same up to coefficients of order unity which we will neglect in the following. The free energy is then given by

$$\beta F = - \int dn_r dm e^{-\beta E} \quad (15)$$

$$= -\beta \int dE dm n_r(E, m) e^{-\beta E} \quad (16)$$

where E is the energy in static patch coordinates, m is the azimuthal quantum number, and $n_r(E, m)$ radial quantum number fixed by

$$n_r = \frac{1}{\pi} \int_L^{l-\epsilon} dr \frac{1}{g(r)} \sqrt{E^2 - \frac{m^2 g(r)}{r^2}}, \quad g(r) = 1 - \frac{r^2}{l^2} \quad (17)$$

for fixed brick-wall cutoff ϵ and infrared cutoff L . We take units where Newton's constant $G_N = 1/8$, and the cosmological constant is $\Lambda = 1/l^2$. In the second line of (15) we have integrated by parts. Usually the analysis proceeds by integrating over m , subject to the condition that the square root in (17) remain positive. Then one is left with an r integral that diverges as $\epsilon \rightarrow 0$, indicating that the states in question are localized near the horizon.

We can write (15) in a more group theoretic way as

$$\beta F = -\text{Tr } e^{-\beta E} \quad (18)$$

where the trace is over states that live in the tensor product of the isometry group $SL(2, \mathbb{C})$ with \mathcal{M} , a spacelike hypersurface through the static patch. The $SL(2, \mathbb{C})$ representation is the unitary principal series, and the trace is restricted to states with positive energy and positive angular momentum, with the brick-wall cutoff left implicit. As shown in [4], the expression (18) arises by taking the Euclidean/Bunch-Davies vacuum, decomposing modes in global coordinates into those on two copies of the static patch, and tracing over modes in the unobserved static patch.

We propose to generalize the expression (18) to the q -deformed case, where the brick-wall cutoff is no longer necessary. Our main motivation for doing this is to illustrate the regularizing properties of the q -deformed theory, in a simple toy model calculation. The full formulation of the second quantized field theory in q -deformed de Sitter will be left for future work. We expect the details of this construction, and the choice of statistics of the fields will change the free energy by coefficients of order unity.

The first step is to perform the analog of the integration over the spacelike surface \mathcal{M} . This should be generated by picking a point in the quantum plane, described in the appendix, and summing over distinct points obtained by acting with the quantum de Sitter isometry group. As discussed in the appendix, this leads to a factor N^2 . The remaining trace over $sl_q(2, \mathbb{C})$ can be directly evaluated using the results of the previous section.

For the Type I (irreducible) representations we find an entropy S_β that is dominated by the $L_0 = 0, \bar{L}_0 = 0$ term for $\beta \gtrsim 2\pi l$. This leads to an entropy approximately equal to

$$S_\infty = N^2 .$$

For the high temperature limit $\beta \rightarrow 0$, where the entropy simply counts the logarithm of the dimension of the entire Hilbert space,

$$S_0 = N^2(N+1)^2/4 .$$

The simplest picture is one where this entropy S_0 is identified with the horizon entropy of de Sitter $S_\Lambda = \pi/2G_N\sqrt{\Lambda}$ [21], allowing N to be fixed in terms of $G_N\sqrt{\Lambda}$. Alternatively, one can identify the entropy $S_{\beta=2\pi l}$ with S_Λ . At this level of analysis, there seems to be no preference to one scheme over the other. In fact, both are at odds with the identification

of the Planck scale with the ultraviolet cutoff that appears in the representation of the quantum group, discussed near the end of section III. Of course, given the crudeness of the identification of entanglement entropy with the gravitational entropy of the de Sitter horizon, this mismatch is not surprising. Our hope is a more refined stringy version of this entropy calculation will improve the situation.

For completeness, let us give the analogous results for entropy S_β of the Type II (reducible) representations. As expected in the low temperature limit,

$$S_0 = N^2$$

due to the sum over points on the spacelike hypersurface. At high temperatures, we simply find the logarithm of the dimension of the Hilbert space,

$$S_\infty = N^2(N^2 + 1)^2/4 .$$

On the other hand, for β of order the Hawking inverse temperature $2\pi l$, there is now a much smoother transition between the two behaviors, since now many modes are relevant in the partition function.

This example of the analog brick wall calculation shows the q -deformation does provide a regulator for the entropy in the quantum deformed de Sitter spacetime. If this emerges as an effective description of string theory in a de Sitter background, we can expect a full q -deformed Virasoro symmetry plays a role. We hope to explore this in future work.

Acknowledgments

The author thanks Alberto Güijosa, Daniel Kabat and Maulik Parikh for valuable discussions. This work is supported in part by DOE grant DE-FE0291ER40688-Task A, NSF U.S.-Mexico Cooperative Research grant #0334379, and US-Israel Binational Science Foundation grant #2000359. The author thanks ICTP, Trieste, Italy for hospitality during the completion of this work.

Appendix: GEOMETRY OF QUANTUM PLANE

This is described in [42], for the case of $U_q(sl(2, \mathbb{C}))$, and we adapt that presentation to the case of interest here. We introduce two complex variables z and \tilde{z} which commute, and

satisfy the relations

$$z^N \sim 1, \quad \tilde{z}^N \sim 1$$

together with a reality condition that sets the complex conjugate of z to \tilde{z} . These variables generate a cyclic representation of $U_q(SL(2, \mathbb{C})_{\mathbb{R}})$ where we identify the generators as

$$X_+ = z^{-1}[z\partial_z + \alpha]_q, \quad X_- = z[\alpha - 1 - z\partial_z]_q, \quad K = q^{N-1-z\partial_z}.$$

Here we introduced the symbol $[x]_q = (q^x - q^{-x})/(q - q^{-1})$, and define α via $ab = \tau(\tau + 1) - \frac{1}{4}(N^2 - 1) = [\alpha]_q^2$. Similar definitions hold for the \bar{X}_{\pm}, \bar{H} , with the replacement $z \rightarrow \tilde{z}$ and $\alpha \rightarrow \tilde{\alpha}$, with $[\tilde{\alpha}]_q^2 = \tilde{\tau}(\tilde{\tau} + 1) - \frac{1}{4}(N^2 - 1)$. The quantum isometry algebra acts on the N^2 -dimensional representation generated by z and \tilde{z} .

A notion of q -derivative may be defined as

$$D_q f(z) = \frac{f(qz) - f(q^{-1}z)}{z(q - q^{-1})} = z^{-1}[z\partial_z]_q f(z).$$

With this definition $D_q z^k = [k]_q z^{k-1}$. Likewise a notion of integration may be defined. This can be regarded as an analog of contour integration

$$\int dz_q f(z) = z(q - q^{-1}) \sum_{j=0}^{N-1} f(q^j z)$$

and is non-vanishing only for terms proportional to z^{-1} .

We interpret the quantum plane as a spacelike hypersurface through quantum de Sitter space, which can be thought of as N^2 points. To see this we consider the orbit of a point under the analog of the two generators of translations in the plane, $X_+ + \bar{X}_+$ and $X_- + \bar{X}_-$. Each of these generators has N distinct eigenvalues, and the individual terms satisfy $X_{\pm}^N \propto 1$, $\bar{X}_{\pm}^N \propto 1$. Arbitrary products of these generators span N^2 linearly independent terms. This corresponds to the statement that isometries acting on a single point can generate a spacelike hypersurface of de Sitter.

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